Developing interactive mathematical visualisations

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In early 2014, researchers at the University of Western Sydney (UWS) developed a suite of interactive mathematical visualisations (IMVs) aimed at improving students' understanding of key concepts in first-level mathematics. This paper outlines the pedagogical and interaction design considerations that informed this development, as well as the processes adopted in preparing the IMVs for classroom use. Guidelines for interaction design that draw on research in human-computer interaction, information visualisation and cognitive technologies are reviewed and contextualised for the specific learning needs of mathematics students at UWS. Examples are given of how these guidelines were factored-in to the IMV development. In addition to the pedagogical and technological dimensions of this work, research-based methods for analysing and evaluating the educational effectiveness of the IMVs are examined. It is expected that these methods will underpin a formative evaluation approach to ongoing design and development of the IMVs.

Keywords: interactive mathematical visualisation, pedagogical and interaction design

Introduction

In late 2013, a group of mathematics educators at the University of Western Sydney initiated a project aimed at improving students' understanding of selected concepts in first-level mathematics. The idea was that perennially difficult topics in calculus and algebra would be presented to students in the form of digital tools that they could interact with in a guided manner to help them make sense of the underlying mathematics (two of these topics and their associated IMVs are introduced in the section 'A pair of IMVs: What is the derivative?; What is the limit?'). The decision to develop learning tools that would allow students to engage these topics in a visual and interactive manner was based on the principle that dynamic mathematical phenomena naturally lend themselves to manipulable, graphical representations. Sedig (2009) notes that other benefits of interactive visualisations include that they: make "latent properties visible, hence amplifying their epistemic utility and extending their communicative power"; provide opportunities for "experimentation and exploration of hypothetical 'what if?' situations"; "guide and transform the path of reasoning and understanding"; and coordinate users' "internal mental models with external visual models of objects and processes" (p. 345). Without explicit reference to interaction, Miller and Upton (2008) note that graphical representations "link more easily with conceptual understanding" (and assert that, by contrast, "symbolic expressions lend themselves to procedural operations") (p. 126); and Kay and Knaack (2007) explain that these representations can "make abstract concepts more concrete" (p. 24, see also Lester, 2000). If done deliberately and with reference to a sound pedagogical and interaction design research base, the addition of interactive elements to graphical illustrations can improve their effectiveness as learning objects (Gadanidis, Sedig, & Liang, 2004; Sedig & Liang, 2006; Sedig, 2009 and references therein).

In this paper we outline the design guidelines and principles that have informed the IMV development, introduce a pair of exemplary IMVs, and consider approaches for their analysis and evaluation.

Pedagogical design

A strong imperative in planning and developing the IMVs was that equal attention be given to the pedagogical and interaction dimensions of their design. While these dimensions are linked in important and instructive ways – e.g. a graphic whose animated illustrations can only be 'stopped' and 'started' would promote learning in a different way from one that possesses an interconnected array of directly manipulable components – it will be clearer to first consider them separately and then draw out their synergies by reference to the two IMVs profiled below.

The pedagogical design of the IMVs relates to those characteristics that specifically support learning. At the macro level, these include the embedding of learning theories and frameworks (for a range of examples see Ertmer & Newby, 1993), while at the micro level they include such factors as content focus and organisation, use of scaffolding, formative assessment, methods of questioning and engagement.

Because of its emphasis on the *creation* – as opposed to *acquisition* – of meaning through interpretation of experience, as well as its embrace of principles such as learner control and the manipulation of information that is 'presented in a variety of different ways' and extensible via the use of problem solving tasks (Ertmer & Newby, 1993, p. 58), constructivism is a useful theoretical model for the learner-centred application of interactive visualisations. Further, as Naylor and Keogh (1999) note, the constructivist notion of learning is an *active* process of knowledge creation in which learners 'construct meaning by linking new ideas with their existing knowledge' (p. 93). With appropriate instructional support and opportunity for socially negotiated, collaborative and 'hands-on' engagement with the IMVs, learners' use of these tools can assist them to make sense of the targeted mathematical concepts in a way that is tied to the context in which these concepts are embedded.

So as to concentrate on specific mathematical topics that first-level university students typically find difficult (see, for example, Asiala, Cottrill, Dubinsky, & Schwingendorf, 1997; Habre & Abboud, 2006; Jordaan, 2005; Williams, 1991), it was decided that the IMVs should be cohesive, self-contained learning objects that represent 'micro-contexts' encapsulating "content and appropriate interactivity to achieve a specific educational goal" (Bradley & Boyle, 2004, p. 373). From a design point of view, this narrow scope means that students can be provided with simple sets of user instructions, contextual/theoretical prompts, and self-learning assessments that guide them towards deep and enduring understanding. Enabling students to control and pace their own learning in a structured and scaffolded manner is also important as there is evidence that for elementary learners free exploration of IMVs 'does not work' (Burgiel, Lieberman, Miller, & Willcox, 2013, p. 132; Edwards & Edwards, 2003; Kay, 2012).

Interaction design

The field of research concerned with the design and operation of human-computer interactive objects is vast. Broad distinctions are drawn between the use of these technologies for productive/entertainment and education/knowledge-specific applications, while at a finer level categories such as representation, presentation, interaction and interactivity are identified and compared (Sedig, 2009). For the purposes of developing an educationally effective suite of IMVs whose design is robust and evidence-based, an account has been taken of various interaction design frameworks. Such frameworks not only provide a coherent or common language for "describing and characterizing the interactive features and properties of (interactive) visualizations", they also

> ...stimulate creativity and innovation in design. Without superimposition of frameworks upon a landscape of design, designers tend to develop tools in an ad hoc fashion, relying mostly on personal intuition and anecdotes, without being able to situate the tools in the landscape. This ad hoc development of tools not only may result in tools that do not address the needs of a situation, but also may have a negative effect on any posterior analysis and evaluation of the effectiveness of the tools (Sedig, 2009, p. 346).

In outlining a number of "influential theories and design principles for informing the design of effective interactive visualisation user interfaces", Hicks (2009) gives a summary of a selection of well-established frameworks (quote from p. 155). These include the nine Gestalt principles of perceptual organisation, the mechanisms of pre-attentive processing, Tufte's principles of 'graphical excellence' (see Tufte, 1983), guidelines for dialogue styles (including, in particular, those that pertain to direct manipulation – see Norman, 1988), and the principles of usability and their associated user experience goals (these are listed in Table 7.2. of Hicks, 2009). Sedig gives a comprehensive description of what he distinguishes as interaction, interactivity and macro interaction frameworks (Sedig, 2009), Miller and Upton highlight several features that governed the design of their 'mathlets' for introductory mathematics students at MIT (Miller & Upton, 2008), and various general references give wide-ranging taxonomies and classifications (see for example Rogers, Sharp, & Preece, 2011; Shneiderman, Plaisant, Cohen, & Jacobs, 2009).

Some of this information was used to frame the design and development of the IMVs under review. For example: the interfaces were kept simple, accessible and task-focussed; multiple representations of mathematical objects were given, and those that were dynamic were made to be directly manipulable; all affordances (visual and textual cues) were added with appropriate standards and encodings; computational/mathematical outputs were made to be scrupulously accurate (and hence amenable to experimental use); pedagogically peripheral yet complex tasks were automated or mechanised (a process referred to as 'cognitive offloading'); and opportunities for comparative investigation of alternative cases were explicitly provided.

A pair of IMVs: What is the derivative? What is the limit?

The two IMVs profiled in this paper cover basic topics in first-level calculus. The first – shown on the left of Figure 1 – allows the user to picture how the derivative is defined as the limit of a sequence of secant gradients. This sequence (which consists of numbers) corresponds to a sequence of lines (the secants) represented as 'ghosts' converging to the unique tangent line. The ghost secants are generated by dragging a moveable secant (shown in blue). Different plots can be selected which show alternative cases, such as where the gradient sequence converges slowly or where no derivative exists. Other features include an adjustable limit direction (left/right) and number of secants, dynamically tabulated secant gradients, tooltips giving secant equations, tangent and anchor/movable point toggles, and dynamic mathematical text (that updates with the moving secant).

The second IMV – shown on the right of Figure 1 – allows the user to explore the so-called δ-ε definition of the limit by adjusting appropriate vertical and horizontal bands (representing open sets). Text to the right of the visualisation – parts of which are dynamically linked to the moving bands – guides the user through the algebraic/symbolic representation of this definition. An alternative plot can be selected which demonstrates the case where no limit exists. When an *x* value is selected inside the vertical band, its decimal representation and location are highlighted, along with those of its corresponding function value.

Both IMVs focus the user on a central graphical interface supporting salient mathematical representations and their relationships, they use colour coding to group different components or alert the user to particular mathematical changes or behaviours, they allow for direct – and reversible – manipulation of key components, and they are 'minimally displayed' (Sedig & Sumner, 2006) in the sense that they are restricted to a single screen on all platforms. The technology used to develop the IMVs is a combination of Java (ported from the Processing language to JavaScript via Processing.js), JavaScript, and HTML5. Various JavaScript libraries – such as JQuery and dat.GUI – have also been used.

In preparing these IMVs for classroom use, experienced mathematical educators were consulted about how best to interweave their pedagogical and interaction design elements. It was agreed that a teacher would induct students into use of the visualisations by guiding their navigation of the interfaces as well as scaffolding their interaction (reasoning, meaning-making) with the mathematical objects represented. Question sets and theoretical prompts would be provided in addition to ungraded (progress-checking) and graded assessments. Students would be asked to work in pairs or groups and communicate their thinking through – and decision-making about – the concepts at hand.

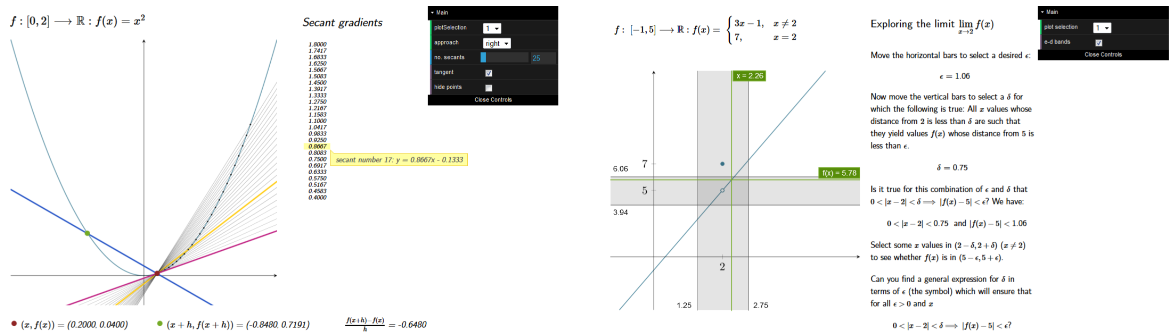

Figure 1: (Left) Screenshot of the What is the derivative? Application showing the moving secant (blue) and a selected 'ghost' secant (yellow). The sequence of 25 secant gradients corresponding to the ghosted secant lines (grey) is seen to converge to the gradient of the tangent (whose corresponding line is pink) to the graph of *f* at the point *(x,f(x))=(0.2,0.04)* – i.e. to the derivative of *f* at *x=0.2*. (Right) Screenshot of the What is the limit? application showing the δ (vertical) and ε (horizontal) bands around *x=2* and *f(x)=5* respectively. A selected *x* value (*x=2.26*) is also shown (green) with its corresponding function value (*f(x)=5.78*) . Numbers in the text to the right of the graph change when the δ and ε bands move.

Analysis and evaluation

Kay (2011) notes that research on the evaluation of web-based learning tools (WBLTs)[1] has lacked experimental and statistical rigour and been limited to technical instructional design issues at the expense of issues related to pedagogy. After an extensive literature review, the author identifies three key constructs for evaluating WBLTs: *learning* (interactivity, good quality feedback, visual supports, whether new concepts have been learned); *design* (clarity of instructions and help features, ease of use, overall organisation and layout); and *engagement* (overall theme, multimedia used, willingness to use WBLTs again). In Kay (2012) these constructs are used to develop "reliable, valid and research-based survey tools… employed to collect data on student attitudes toward learning objects" (p. 354, and noting that 'learning objects' are defined in exactly the same way as WBLTs). Five research questions addressing student attitudes toward mathematics-based learning objects, their performance as a result of using these objects and cross-tabulations with student characteristics, 'instructional architecture' and teaching strategy, give a clear structure for the author's evaluative investigations.

These ideas will assist in the formulation of a framework and methodology for the evaluation of the IMVs.

Conclusion

The decision to design and develop a suite of IMVs for first-level students at the University of Western Sydney was motivated by a desire to make inherently dynamic mathematical representations more engaging and conceptually accessible. The development itself has been informed by the growing research base in pedagogical and interaction design; and the analysis and evaluation of the technical and learning-related performance of the IMVs will be informed by the experience of other researchers in closely related areas of inquiry.

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[1] Defined as 'interactive web-based tools that support the learning of specific concepts by enhancing, amplifying, and/or guiding the cognitive processes of learners' (Kay, 2011, p. 1849).

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Please cite as: Pettigrew, J., & Shearman, D. (2014). Developing interactive mathematical visualisations. In B. Hegarty, J. McDonald, & S.-K. Loke (Eds.), *Rhetoric and Reality: Critical perspectives on educational technology. Proceedings ascilite Dunedin 2014* (pp. 539-543).

Note: All published papers are refereed, having undergone a double-blind peer-review process.